Lie Superalgebras and the Multiplet Structure of the Genetic Code I: Codon Representations

Michael Forger and Sebastian Sachse [1]

Departamento de Matemática Aplicada,
Instituto de Matemática e Estatística,
Universidade de São Paulo,
Cx. Postal 66281, BR–05315-970 São Paulo, S.P., Brazil

February 4, 2008

Abstract

It has been proposed [1] that the degeneracy of the genetic code, i.e., the phenomenon that different codons (base triplets) of DNA are transcribed into the same amino acid, may be interpreted as the result of a symmetry breaking process. In ref. [1] this picture was developed in the framework of simple Lie algebras. Here, we explore the possibility of explaining the degeneracy of the genetic code using basic classical Lie superalgebras, whose representation theory is sufficiently well understood, at least as far as typical representations are concerned. In the present paper, we give the complete list of all typical codon representations (typical 64-dimensional irreducible representations), whereas in the second part, we shall present the corresponding branching rules and discuss which of them reproduce the multiplet structure of the genetic code.

[1] Work supported by FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo) and CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Brazil

1 Introduction

The discovery of the molecular structure of DNA by Watson and Crick in 1953 was the most important step towards an understanding of the physiological basis for the storage and transfer of genetic information. DNA is a macromolecule in the form of a double helix which encodes this information in a language with 64 three-letter words built from an alphabet with a set of four different letters (the four nucleic bases attached to the backbone of a DNA molecule). These words are called codons and form sentences called genes. Each codon can be translated into one of twenty amino acids or a termination signal. This leads to a degeneracy of the code in the sense that different codons represent the same amino acid, that is, different words have the same meaning. In fact, the codons which code for the same amino acids form multiplets as follows:

- 3 sextets — Arg, Leu, Ser
- 5 quadruplets — Ala, Gly, Pro, Thr, Val
- 2 triplets — Ile, Term
- 9 doublets — Asn, Asp, Cys, Gln, Glu, His, Lys, Phe, Tyr
- 2 singlets — Met, Trp

When a protein is synthesized, an appropriate segment of one of the two strings in the DNA molecule (or more precisely, the mRNA molecule built from it) is read and the corresponding amino acids are assembled sequentially. The linear chain thus obtained will then fold to the final configuration of the protein.

These well-known facts, however, provide no explanation as to why just this special language has been chosen by nature. Since its discovery, the genetic code has essentially remained a table connecting codons (base triplets) with the amino acids they represent, but a complete understanding of its structure is still missing.

A new approach to the question was suggested in 1993 by Hornos & Hornos [1] who proposed to explain the degeneracy of the genetic code as the result of a symmetry breaking process. The demand of this approach can be compared to explaining the arrangement of the chemical elements in the periodic table as the result of an underlying dynamical symmetry which is reflected in the electronic shell structure of atoms. Another comparable example is the explanation of the multiplet structure of hadrons as a result of a "flavor" $SU(3)$ symmetry, which led to the quark model and to the prediction of new particles. An interesting and important feature of this "flavor" symmetry is its internal or dynamical nature, that is, it is an internal property of the dynamical equations of the system, rather than being related to the structure of space-time.

In the same spirit, the idea of the above mentioned authors was to explain the multiplet structure of the genetic code through the multiplets found in the codon representation (= irreducible 64-dimensional representation) of an appropriate simple Lie algebra and its branching rules into irreducible representations of its semisimple subalgebras. They checked the tables of branching rules of McKay and Patera [2] for semisimple subalgebras of simple Lie algebras of rank ≤ 8. The most suitable multiplet structure found is derived from the codon representation of the symplectic algebra $\mathfrak{sp}(6)$ by the following sequence of symmetry breakings:

$$\begin{array}{rcll} \mathfrak{sp}(6) & \supset & \mathfrak{sp}(4) \oplus \mathfrak{su}(2) & \qquad I \\ & \supset & \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) & \qquad II \\ & \supset & \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) & \qquad III/IV/V \end{array}$$

The sequence of steps $I - V$ is interpreted as the evolution of the genetic code in the early time of organic life.

This work, which had a strong resonance in the scientific community [3, 4, 5], raised a lot of new interesting problems. One of these is that the last step in the symmetry breaking is incomplete: the lifting of degeneracy by breaking the last two $\mathfrak{su}(2)$ subalgebras to $\mathfrak{u}(1)$ is not followed by all codon multiplets. Only if some of them continue to represent a single amino acid can the actual multiplet structure of the genetic code be obtained. This "freezing" had already been proposed by biologists [6] who claimed that a completely accomplished evolution of the genetic code should have resulted in 28 amino acids [7] (for a more recent review including an extensive bibliography, see [8]) – in perfect agreement with the mathematical model under consideration. However the phenomenon that some of the multiplets preserve a symmetry while it is broken in others, even though it does not contradict a purely biological theory (in fact biologists wonder why there should be a mathematical theory at all), is quite awkward from a mathematical point of view.

The basic idea behind the present project, already proposed in [1], is to investigate the "vicinity" of ordinary Lie algebras, namely quantum groups and Lie superalgebras. As it turns out, the main new problem which appears in this context, both for quantum groups and for Lie superalgebras, is the existence of indecomposable representations, i.e., representations which are reducible but not fully reducible: they contain irreducible subrepresentations but cannot be decomposed into the direct sum of irreducible subrepresentations. As a result, the representation theory of quantum groups and of Lie superalgebras is not developed to the same extent as that of ordinary (reductive) Lie algebras. Therefore, the task of performing an exhaustive search is presently not feasible: one may at best hope for partial results. Some steps in this direction have recently been taken by various authors [9, 10, 11].

2 Basic classical Lie superalgebras

We begin by recalling that a *Lie superalgebra* (LSA) is a $\mathbb{Z}_2$-graded vector space

$$\mathfrak{g} \;=\; \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \tag{1}$$

equipped with a bilinear map $[.\,,.] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ called the *supercommutator* which is *homogeneous* of degree 0 (i.e., satisfies $\deg([X,Y]) = \deg(X) + \deg(Y)$ for homogeneous $X, Y \in \mathfrak{g}$), is *graded antisymmetric*,

$$[Y,X] \;=\; -\,(-1)^{\deg(X)\deg(Y)}\,[X,Y] \qquad \text{for homogeneous } X,Y \in \mathfrak{g}\ ,$$

and satisfies the *graded Jacobi identity*,

$$\begin{aligned} (-1)^{\deg(X)\deg(Z)}\,[X,[Y,Z]] \,+\, (-1)^{\deg(X)\deg(Y)}\,[Y,[Z,X]] & \\ +\,(-1)^{\deg(Y)\deg(Z)}\,[Z,[X,Y]] \;=\; 0 & \end{aligned}$$

$$\text{for homogeneous } X,Y,Z \in \mathfrak{g}\ .$$

In particular, the even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ is an ordinary Lie algebra and the odd part $\mathfrak{g}_{\bar{1}}$ of $\mathfrak{g}$ carries a representation of $\mathfrak{g}_{\bar{0}}$, i.e., is a $\mathfrak{g}_{\bar{0}}$-module. In the following we shall be dealing exclusively with finite-dimensional complex Lie superalgebras which are *simple*, i.e., admit no non-trivial ideals. Such a Lie superalgebra is called *classical* if its even part $\mathfrak{g}_{\bar{0}}$ is reductive, that is, if it decomposes into the direct sum of its center and a semisimple subalgebra, or equivalently, if all representations of $\mathfrak{g}_{\bar{0}}$ (in particular that on $\mathfrak{g}_{\bar{0}}$ itself, which is the adjoint representation, and that on $\mathfrak{g}_{\bar{1}}$) are completely reducible [12, 13, 14, 15]. Note that this property is not guaranteed automatically, as it would be for ordinary semisimple Lie algebras, according to Weyl's theorem. (However, the term "classical" in this context is unfortunate because it suggests that "classical" for simple Lie superalgebras bears some relation to the standard term "classical", in the sense of "non-exceptional", for simple Lie algebras, which is not the case.) A standard argument then shows [14, 15] that a classical Lie superalgebra necessarily belongs to one of the following two types:

- *Type I*:

  The representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is the direct sum of two mutually conjugate irreducible representations,

  $$\mathfrak{g}_{\bar{1}} \;=\; \mathfrak{g}_{1} \oplus \mathfrak{g}_{-1}\ . \tag{2}$$

  We distinguish two subcases:

  - *Type I_0*:

    The center $\mathfrak{z}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$ is trivial, i.e., $\mathfrak{g}_{\bar{0}}$ is semisimple.

  - *Type I_1*:

    The center $\mathfrak{z}_{\bar{0}}$ of $\mathfrak{g}_{\bar{0}}$ is non-trivial. In this case, $\mathfrak{z}_{\bar{0}}$ is one-dimensional and is generated by an element c which – when appropriately normalized – acts as the identity on $\mathfrak{g}_1$ and as minus the identity on $\mathfrak{g}_{-1}$.

- *Type II*:

  The representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is irreducible. In this case, the center of $\mathfrak{g}_{\bar{0}}$ is necessarily trivial, or in other words, $\mathfrak{g}_{\bar{0}}$ is semisimple.

Another important concept for the analysis and classification of simple Lie superalgebras is the question whether they admit non-degenerate invariant forms. Recall that a bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is called *even* if it is homogeneous of degree 0 (i.e., satisfies $B(X,Y) = 0$ if $X \in \mathfrak{g}_{\bar{0}}$ and $Y \in \mathfrak{g}_{\bar{1}}$ or $X \in \mathfrak{g}_{\bar{1}}$ and $Y \in \mathfrak{g}_{\bar{0}}$), is called *graded symmetric* if

$$B(Y,X) \;=\; (-1)^{\deg(X)\deg(Y)}\, B(X,Y) \qquad \text{for homogeneous } X,Y \in \mathfrak{g}\;,$$

and is called *invariant* if

$$B([X,Y],Z) \;=\; B(X,[Y,Z]) \qquad \text{for homogeneous } X,Y,Z \in \mathfrak{g}\;.$$

A simple Lie superalgebra is called *basic* if it admits an even, graded symmetric, invariant bilinear form which is non-degenerate. Note, again, that this property is not guaranteed automatically, as it would be for ordinary semisimple Lie algebras, according to Cartan's criterion for semisimplicity. In fact, it turns out that an even, graded symmetric, invariant bilinear form on a simple Lie superalgebra is either non-degenerate or identically zero [14, 15] and that, in particular, the Killing form of a simple Lie superalgebra defined by the supertrace operation in the adjoint representation may vanish identically. Moreover, there are simple Lie superalgebras whose Killing form vanishes identically but which are still basic because they admit some other non-degenerate, even, graded symmetric, invariant bilinear form.

The structure theory of basic classical Lie superalgebras is to some extent analogous to that of ordinary semisimple Lie algebras. The first step is to choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which is by definition just a Cartan subalgebra of its even part $\mathfrak{g}_{\bar{0}}$: its dimension is called the *rank* of $\mathfrak{g}$. (If $\mathfrak{g}_{\bar{0}}$ has a non-trivial center $\mathfrak{z}_0$ and a semisimple part $\mathfrak{g}_{\bar{0}}^{\mathrm{ss}}$, so that $\mathfrak{g}_{\bar{0}} = \mathfrak{z}_{\bar{0}} \oplus \mathfrak{g}_{\bar{0}}^{\mathrm{ss}}$, then $\mathfrak{h} = \mathfrak{z}_{\bar{0}} \oplus \mathfrak{h}^{\mathrm{ss}}$, where $\mathfrak{h}^{\mathrm{ss}}$ is a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}^{\mathrm{ss}}$.) As in the case of ordinary semisimple Lie algebras, the specific choice of Cartan subalgebra is irrelevant, since they are all conjugate [14, 15]. This gives rise to the *root system* $\Delta = \Delta_0 \cup \Delta_1$ of $\mathfrak{g}$, where the set Δ_0 of *even roots* is just the root system of $\mathfrak{g}_{\bar{0}}$, as an ordinary reductive Lie algebra, and the set Δ_1 of *odd roots* is just the weight system of $\mathfrak{g}_{\bar{1}}$, as a $\mathfrak{g}_{\bar{0}}$-module. Again

as in the case of ordinary semisimple Lie algebras, one associates to each root $\alpha \in \Delta$ a unique generator $H_\alpha \in \mathfrak{h}$, defined by

$$B(H_\alpha, H) \;=\; \alpha(H) \qquad \text{for all } H \in \mathfrak{h} \;,$$

puts

$$(\alpha, \beta) \;=\; B(H_\alpha, H_\beta) \qquad \text{for } \alpha, \beta \in \Delta \;,$$

and considers the real subspace $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}$ formed by linear combinations of the H_α with real coefficients. However, the restriction of the invariant form B to $\mathfrak{h}_{\mathbb{R}}$, which in the case of ordinary semisimple Lie algebras is positive definite when B is chosen to be the Killing form, may now be indefinite since even in those cases where the Killing form of $\mathfrak{g}$ is non-degenerate, its restriction to the simple ideals in $\mathfrak{g}_{\bar{0}}$ (in most cases, there are precisely two such simple ideals) will on one of these simple ideals be a positive multiple of its Killing form but on the other one be a negative multiple of its Killing form, so that even roots α will satisfy either $(\alpha, \alpha) > 0$ or $(\alpha, \alpha) < 0$ whereas odd roots α will in many cases be isotropic: $(\alpha, \alpha) = 0$. Even worse: when the Killing form of $\mathfrak{g}$ vanishes identically, it may happen that B cannot be chosen to take only real values on $\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}$.

This unusual kind of geometry is responsible for various complications that arise in the next steps, which are the choice of an ordering in Δ, corresponding to the choice of a system of *simple roots* α_i $(1 \leq i \leq r)$, the definition of the *Cartan matrix* a and the classification of the basic classical Lie superalgebras in terms of *Kac-Dynkin diagrams*. To begin with, not all orderings are equivalent: different choices may lead to different diagrams. To remove this kind of ambiguity, it is convenient to restrict the allowed orderings to a specific class, corresponding to a *distinguished choice* of simple roots, characterized by the fact that there is only one simple root which is odd, whereas the remaining ones are even. As an example, consider the class of basic classical Lie superalgebras $\mathfrak{g}$ of type I_1 (see above): here, the simple even roots are the simple roots of $\mathfrak{g}_{\bar{0}}$, extended to take the value 0 on c, whereas the simple odd root is minus the highest weight of $\mathfrak{g}_1$, as an irreducible $\mathfrak{g}_{\bar{0}}$-module, which takes the value 1 on c. In general, any such ordering gives rise to a Cartan-Weyl decomposition

$$\mathfrak{g} \;=\; \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \;, \tag{3}$$

where $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are the nilpotent subalgebras spanned by the generators corresponding to positive and negative roots, respectively. Combining this with the direct decomposition (1), one arrives at the distinguished $\mathbb{Z}$-gradation of $\mathfrak{g}$,

$$\begin{array}{rcll} \mathfrak{g} & = & \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} & \text{for type I} \\ \mathfrak{g} & = & \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} & \text{for type II} \end{array} \;, \tag{4}$$

where

$$\mathfrak{g}_1 \;=\; \mathfrak{g}_{\bar{1}} \cap \mathfrak{n}^+ \qquad \text{and} \qquad \mathfrak{g}_{-1} \;=\; \mathfrak{g}_{\bar{1}} \cap \mathfrak{n}^- \tag{5}$$

are spanned by generators corresponding to positive and negative odd roots, respectively, whereas

$$\mathfrak{g}_2 \;=\; [\mathfrak{g}_1, \mathfrak{g}_1] \qquad \text{and} \qquad \mathfrak{g}_{-2} \;=\; [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] \tag{6}$$

are spanned by generators that can be written as anticommutators of these. (Nonvanishing anticommutators of this kind exist only for basic classical Lie superalgebras of type II.) The simple roots are linearly independent, and their number r is equal to the rank of $\mathfrak{g}$, except for the basic classical Lie superalgebras $\mathfrak{g}$ of type I_0, where the simple roots are subject to one linear relation, so their number r exceeds the rank of $\mathfrak{g}$ by 1. The definition of the Cartan matrix a must also be modified, due to the possible occurence of simple odd roots of length 0. When $(\alpha_i, \alpha_i) \neq 0$, one puts

$$a_{ij} \;=\; \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\;,$$

as usual, whereas if $(\alpha_i, \alpha_i) = 0$, one defines

$$a_{ij} \;=\; \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_{i'})}\;,$$

where i' is an appropriately chosen index such that $(\alpha_i, \alpha_{i'}) \neq 0$, whose precise definition is partly a matter of convention. With a distinguished choice of simple roots, this can only happen for the unique simple odd root, i.e., when $i = s$, and the numbering of simple roots is then arranged in such a way that either $i' = s + 1$ or $i' = s - 1$. In this way, $\mathfrak{g}$ is, up to isomorphism, determined by its Cartan matrix, being generated by r positive generators $e_i \in \mathfrak{n}^+$ and r negative generators $f_i \in \mathfrak{n}^-$ satisfying the supercommutation relations

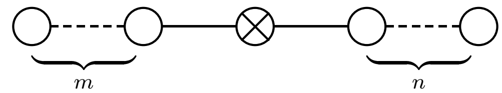
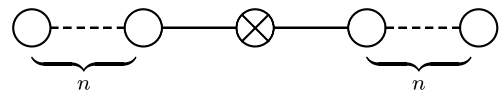
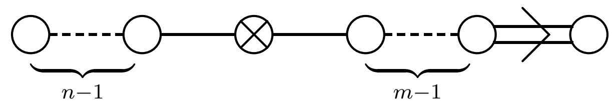
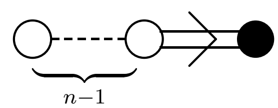
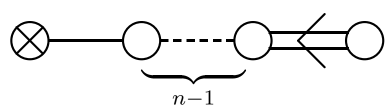
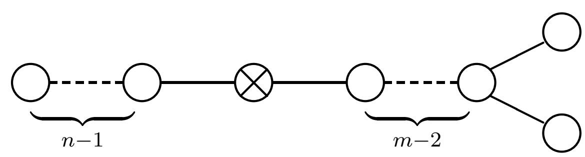
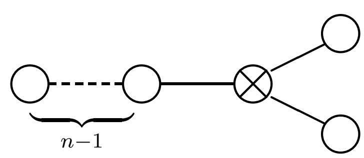
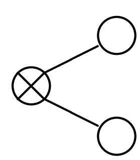
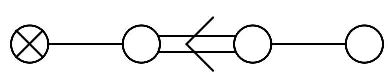
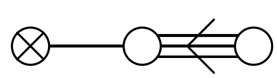
$$[e_i, f_j] \;=\; \delta_{ij} h_i \quad,\quad [h_i, h_j] \;=\; 0 \quad,\quad [h_i, e_j] \;=\; a_{ij}\, e_j \quad,\quad [h_i, f_j] \;=\; -\, a_{ij}\, f_j$$

(plus Serre relations that we do not write down). Finally, the Kac-Dynkin diagram associated with $\mathfrak{g}$ is drawn according to the following rules:

- Simple even roots α_i are denoted by white blobs ○, while the unique simple odd root α_s is denoted by a crossed blob $\otimes$ if it has zero length and by a black blob ● if it has non-zero length.
- The j^{th} and k^{th} simple root are connected by $\max\{|a_{jk}|, |a_{kj}|\}$ lines, except for the Lie superalgebras $D(2\,|\,1; \alpha)$, where the simple odd root is connected to each of the two simple even roots by a single line.
- When the j^{th} and k^{th} simple root are connected by more than a single line, an arrow is drawn pointing from the longer one to the shorter one.

The Kac-Dynkin diagrams of all basic classical Lie superalgebras are listed in the following table.

Table 1: Kac-Dynkin diagrams of the basic classical Lie superalgebras

LSA	Type	Diagram
$A(m\,\vert\,n) \;=\; \mathfrak{sl}(m+1\,\vert\,n+1)$ $(m > n \geq 0)$	I_1	
$A(n\,\vert\,n) \;=\; \mathfrak{sl}(n+1\,\vert\,n+1)/\langle 1 \rangle$ $(n \geq 1)$	I_0	
$B(m\,\vert\,n) = \mathfrak{osp}(2m+1\,\vert\,2n)$ $(m, n \geq 1)$	II	
$B(0\,\vert\,n) = \mathfrak{osp}(1\,\vert\,2n)$ $(n \geq 1)$	II	
$C(n+1) = \mathfrak{osp}(2\,\vert\,2n)$ $(n \geq 1)$	I_1	
$D(m\,\vert\,n) = \mathfrak{osp}(2m\,\vert\,2n)$ $(m \geq 3, n \geq 1)$	II	
$D(2\,\vert\,n) = \mathfrak{osp}(4\,\vert\,2n)$ $(n \geq 1)$	II	
$D(2\,\vert\,1;\alpha) = \mathfrak{osp}(4\,\vert\,2\,;\alpha)$ $(\alpha \neq 0, -1, \infty)$	II	
$F(4)$	II	
$G(3)$	II	

Observe that the Cartan matrix cannot always be reconstructed uniquely from the corresponding Kac-Dynkin diagram, in particular this happens for the Lie superalgebras $D(2\,|\,1;\alpha)$.

The basic classical Lie superalgebras of type I_0 are in many respects pathological, but almost all the general results about basic classical Lie superalgebras (including the main ones from representation theory) remain true if one replaces $A(n\,|\,n) = \mathfrak{sl}(n+1\,|\,n+1)/\langle 1\rangle$ by its natural central extension $\mathfrak{sl}(n+1\,|\,n+1)$. (This leads, for example, to an enrichment of the representation theory, since the irreducible representations of the former form a subclass of the irreducible representations of the latter: namely those in which the central element is represented by the zero operator.) We shall therefore, throughout the rest of this paper, adopt the following terminology:

- Type I Lie superalgebras:
  $\mathfrak{sl}(p\,|\,q)$ with $p \geq q \geq 1$ and $(p,q) \neq (1,1)$
  (the case $p = q = 1$ is excluded since $A(0,0)$ is not simple),
  $\mathfrak{osp}(2\,|\,2n)$ with $n \geq 1$ and $n \neq 1$
  (the case $n = 1$ is excluded since $\mathfrak{osp}(2\,|\,2) \cong \mathfrak{sl}(3\,|\,2)$).

- Type II Lie superalgebras:
  $\mathfrak{osp}(p\,|\,2n)$ with $p = 1$ or $p \geq 3$ and $n \neq 1$,
  $\mathfrak{osp}(4\,|\,2\,;\alpha)$, $F(4)$, $G(3)$.

It is also interesting to compare the Kac-Dynkin diagram of $\mathfrak{g}$ with the Dynkin diagram of its even part $\mathfrak{g}_{\bar{0}}$ and the Dynkin diagram of the subalgebra $\mathfrak{g}_0$ that appears in the direct decomposition (3). For type I Lie superalgebras, where $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$, the latter is obtained from the former by simply removing the simple odd root α_s, which may therefore be thought of as representing the one-dimensional center of the even part, whereas for type II Lie superalgebras, the Dynkin diagram of $\mathfrak{g}_0$ is obtained from the Kac-Dynkin diagram of $\mathfrak{g}$ by removing the simple odd root α_s and from the Dynkin diagram of $\mathfrak{g}_{\bar{0}}$ by removing one of its simple roots: this simple root, which we shall denote by α_s^0, is usually referred to as the "hidden" simple root of $\mathfrak{g}_{\bar{0}}$ because in the Kac-Dynkin diagram of $\mathfrak{g}$, it can be thought of as being "hidden behind" the simple odd root α_s.

3 Representation theory

The representation theory of basic classical Lie superalgebras $\mathfrak{g}$ (with $A(n\,|\,n) = \mathfrak{sl}(n+1\,|\,n+1)/\langle 1\rangle$ replaced by $\mathfrak{sl}(n+1\,|\,n+1)$; see above) has been developed by

Kac [12, 13]. Using the Poincaré-Birkhoff-Witt theorem, the finite-dimensional irreducible representations of $\mathfrak{g}$ are constructed by the method of induced representations, that is, as quotient spaces of Verma modules by their maximal invariant subspaces. This implies that all finite-dimensional irreducible representations of $\mathfrak{g}$ are highest weight representations, that is, representations of the form $\pi_\Lambda : \mathfrak{g} \to \mathrm{End}(V_\Lambda)$ associated to a highest weight $\Lambda \in \mathfrak{h}^*$ and characterized by the presence of a non-zero cyclic vector $v_\Lambda \in V_\Lambda$ satisfying

$$\mathfrak{n}^+(v_\Lambda) \;=\; 0 \qquad \text{and} \qquad H(v_\Lambda) \;=\; \Lambda(H)\, v_\Lambda \quad \text{for all } \; H \in \mathfrak{h} \; .$$

A necessary condition for such a representation to be finite-dimensional is that Λ is *dominant integral*, which means that the Dynkin labels

$$l_i \;=\; \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \tag{7}$$

of Λ associated with the simple even roots α_i $(i = 1, \ldots, r,\, i \neq s)$ of $\mathfrak{g}$ must be non-negative integers. For type I Lie superalgebras, this is the only condition to be imposed. In particular, the value of the Dynkin label

$$l_s \;=\; \frac{(\Lambda, \alpha_s)}{(\alpha_s, \alpha_{s'})} \tag{8}$$

of Λ associated with the simple odd root α_s of $\mathfrak{g}$ may in this case be an arbitrary complex number, whereas for type II Lie superalgebras, it is subject to additional restrictions: some of these simply express the requirement that the Dynkin label

$$l_s^0 \;=\; \frac{2(\Lambda, \alpha_s^0)}{(\alpha_s^0, \alpha_s^0)} \tag{9}$$

of Λ associated with the hidden simple root α_s^0 of $\mathfrak{g}_{\bar{0}}$ must also be a non-negative integer, while the others are supplementary conditions to guarantee that Λ is the highest weight of a finite-dimensional irreducible representation not only of $\mathfrak{g}_{\bar{0}}$ but also of $\mathfrak{g}$. For detailed formulae, see [15, 12, 13, 16].

An explicit construction of the representation $\pi_\Lambda : \mathfrak{g} \to \mathrm{End}(V_\Lambda)$ of $\mathfrak{g}$ starts out from the representation $\pi_{\Lambda,0} : \mathfrak{g}_0 \to \mathrm{End}(V_{\Lambda,0})$ of $\mathfrak{g}_0$ with highest weight Λ, or more precisely, with highest weight given by the restriction of Λ to the intersection of $\mathfrak{g}_0$ with the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This representation is first extended to a representation of the subalgebra $\mathfrak{k} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ by letting $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ act trivially on $V_{\Lambda,0}$. Then define

$$\begin{aligned} \bar{V}_\Lambda \;&=\; \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}\, V_{\Lambda,0} && \text{for type I Lie superalgebras ,} \\ \bar{V}_\Lambda \;&=\; \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}\, V_{\Lambda,0}\,/\,M && \text{for type II Lie superalgebras ,} \end{aligned} \tag{10}$$

where the invariant submodule M is obtained by applying arbitrary linear combinations of products of elements of $\mathfrak{g}$ (i.e., the enveloping algebra $U(\mathfrak{g})$) to the vector obtained by (l_s^0+1)-fold application of the even generator $E_{-\alpha_s^0} \in \mathfrak{g}_{-2}$ to the highest weight vector v_Λ:

$$M \;=\; \langle\, U(\mathfrak{g})\, E_{-\alpha_s^0}^{l_s^0+1} v_\Lambda \rangle \;.$$

The Kac module $\bar{V}_\Lambda$ is finite-dimensional and contains a unique maximal submodule $\bar{I}_\Lambda$. Then

$$V_\Lambda \;=\; \bar{V}_\Lambda \,/\, \bar{I}_\Lambda \;. \tag{11}$$

Any finite-dimensional irreducible representation of $\mathfrak{g}$ can be obtained in this way. However, it is in general difficult to gain control over the submodule $\bar{I}_\Lambda$, so explicit calculations are usually only possible when this submodule vanishes – which is one of the main reasons for the special role played by the so-called *typical representations*:

$$\bar{I}_\Lambda \;=\; \{0\} \quad,\quad V_\Lambda \;=\; \bar{V}_\Lambda \qquad \text{for typical representations}$$

Typical representations are, by definition, irreducible representations that may appear as direct summands in completely reducible representations only, whereas irreducible representations appearing as subrepresentations of indecomposable (that is, reducible but not completely reducible) representations are called atypical. A useful criterion for an irreducible representation to be typical is that $(\Lambda+\rho,\alpha)\neq 0$ for all odd roots α for which 2α is not an even root, where

$$\rho \;=\; \rho_0 - \rho_1 \quad,\quad \rho_0 \;=\; \tfrac{1}{2}\sum_{\alpha\in\Delta_0^+}\alpha \quad,\quad \rho_1 \;=\; \tfrac{1}{2}\sum_{\alpha\in\Delta_1^+}\alpha$$

Denoting the number of positive odd roots, i.e., the cardinality of Δ_1^+, by N_1 (and similarly, the number of positive even roots, i.e., the cardinality of Δ_0^+, by N_0), one can write down an explicit formula for the total dimension of any typical representation:

$$\dim V_\Lambda \;=\; 2^{N_1} \prod_{\alpha\in\Delta_0^+} \frac{(\Lambda+\rho,\alpha)}{(\rho_0,\alpha)} \;. \tag{12}$$

This formula can be simplified by expressing the product on the rhs in terms of the standard Weyl dimension formula for an irreducible representation $\pi_{\tilde{\Lambda},\bar{0}} : \mathfrak{g}_{\bar{0}} \rightarrow \mathrm{End}(V_{\tilde{\Lambda},\bar{0}})$ of the even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ with highest weight $\tilde{\Lambda}$:

$$\dim V_{\tilde{\Lambda},\bar{0}} \;=\; \prod_{\alpha\in\Delta_0^+} \frac{(\tilde{\Lambda}+\rho_0,\alpha)}{(\rho_0,\alpha)} \;. \tag{13}$$

To establish the desired relation, observe that $(\rho_1,\alpha_i) = 0$ for all simple even roots α_i of $\mathfrak{g}$ because the corresponding positive and negative root generators

E_{α_i} and $E_{-\alpha_i}$ belong to $\mathfrak{g}_0$ and hence preserve the subspaces in the direct decomposition (4): therefore, the number $(2\rho_1, \alpha_i)$, which is precisely the trace of the operator on $\mathfrak{g}_1$ that represents $H_{\alpha_i} = [E_{\alpha_i}, E_{-\alpha_i}]$, must vanish. Therefore, for type I Lie superalgebras, we may simply put $\tilde{\Lambda} = \Lambda$, so

$$\dim V_\Lambda \;=\; 2^{N_1} \dim V_{\Lambda,\bar{0}} \;. \tag{14}$$

An alternative argument for deriving this formula is to use the construction of the Kac module because, in this case, $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$, $V_{\Lambda,\bar{0}} = V_{\Lambda,0}$ and $[E_\alpha, E_\beta] = 0$ for all positive odd roots $\alpha, \beta \in \Delta_1^+$, so that

$$V_\Lambda \;=\; \bar{V}_\Lambda \;=\; \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}} V_{\Lambda,0} \;\cong\; \bigwedge \mathfrak{g}_{-1} \otimes V_{\Lambda,0} \;,$$

where $\bigwedge \mathfrak{g}_{-1}$ denotes the exterior or Grassmann algebra over $\mathfrak{g}_{-1}$, which has dimension 2^{N_1}. For type II Lie superalgebras, we let $\{\lambda_1, \ldots, \lambda_{s-1}, \lambda_s^0, \lambda_{s+1}, \ldots, \lambda_r\}$ denote the basis of fundamental weights dual to the basis $\{\alpha_1, \ldots, \alpha_{s-1}, \alpha_s^0, \alpha_{s+1}, \ldots, \alpha_r\}$ of simple roots for $\mathfrak{g}_{\bar{0}}$ and introduce the shifted highest weight

$$\tilde{\Lambda} \;=\; \Lambda - \frac{2(\rho_1, \alpha_s^0)}{(\alpha_s^0, \alpha_s^0)} \, \lambda_s^0 \;, \tag{15}$$

which in terms of Dynkin labels means

$$\tilde{l}_i = l_i \quad \text{for} \quad i \neq s \quad , \quad \tilde{l}_s^0 \;=\; l_s^0 - \frac{2(\rho_1, \alpha_s^0)}{(\alpha_s^0, \alpha_s^0)} \;. \tag{16}$$

It should be noted that although the original highest weight Λ is dominant integral, the shifted highest weight $\tilde{\Lambda}$ need not be, since $2(\rho_1, \alpha_s^0)/(\alpha_s^0, \alpha_s^0)$ may assume half-integer values (see Table 2), so $\tilde{l}_s^0$ may become half-integer and/or negative. In this case, equation (13) is only formal, in the sense that the expression "$\dim V_{\tilde{\Lambda},\bar{0}}$" does not necessarily stand for the dimension of an irreducible representation of $\mathfrak{g}_{\bar{0}}$. Therefore, we introduce for every ordinary semisimple Lie algebra $\mathfrak{a}$ of rank p the abbreviation $d_{\mathfrak{a}}$ to denote the dimension function for its irreducible representations, which is a polynomial in p variables given by the standard Weyl dimension formula, and we simply write $d_{\bar{0}}$ instead of $d_{\mathfrak{g}_{\bar{0}}}$, so equation (13) is replaced by

$$d_{\bar{0}}(\tilde{\Lambda}) \;=\; \prod_{\alpha \in \Delta_0^+} \frac{(\tilde{\Lambda} + \rho_0, \alpha)}{(\rho_0, \alpha)} \;. \tag{17}$$

Then equation (12) becomes

$$\dim V_\Lambda \;=\; 2^{N_1} \, d_{\bar{0}}(\tilde{\Lambda}) \;. \tag{18}$$

In order to proceed further, we need more information on the behavior of the function $d_{\bar{0}}$. First of all, we observe that as long as

$$l_s^0 \geq b \qquad \text{i.e.} \qquad \tilde{l}_s^0 \;\geq\; -\tfrac{1}{2} \;, \tag{19}$$

Table 2: Shift of highest weight for type II Lie superalgebras

LSA	$\dfrac{2(\rho_1,\alpha_s^0)}{(\alpha_s^0,\alpha_s^0)}$	b
$B(m\,\|\,n) = \mathfrak{osp}(2m+1\,\|\,2n)$ $(m,n \geq 1)$	$m+\frac{1}{2}$	m
$B(0\,\|\,n) = \mathfrak{osp}(1\,\|\,2n)$ $(n \geq 1)$	$\frac{1}{2}$	0
$D(m\,\|\,n) = \mathfrak{osp}(2m\,\|\,2n)$ $(m \geq 2, n \geq 1)$	m	m
$D(2\,\|\,1;\alpha) = \mathfrak{osp}(4\,\|\,2\,;\alpha)$ $(\alpha \neq 0,-1,\infty)$	2	2
$F(4)$	4	4
$G(3)$	$\frac{7}{2}$	3

where b is the integer part of $2(\rho_1,\alpha_s^0)/(\alpha_s^0,\alpha_s^0)$ (see Table 2), all factors in the product on the rhs of equation (17) remain positive. Hence in this region, $d_{\bar{0}}$ is positive and monotonically increasing in the following sense: Suppose that $\tilde{\Lambda}$ and $\tilde{M}$ are two highest weights for $\mathfrak{g}_{\bar{0}}$, with Dynkin labels l_i, $\tilde{l}_s^0$ and m_i, $\tilde{m}_s^0$, respectively, where $i = 1,\ldots,r$, $i \neq s$ and $\tilde{l}_s^0, \tilde{m}_s^0 \geq -\frac{1}{2}$. Then defining

$$\tilde{\Lambda} \geq \tilde{M} \qquad \Longleftrightarrow \qquad l_i \geq m_i \ \ (1 \leq i \leq r,\ i \neq s) \ \text{ and } \ \tilde{l}_s^0 \geq \tilde{m}_s^0 \ , \tag{20}$$

and $\tilde{\Lambda} > \tilde{M}$ iff $\tilde{\Lambda} \geq \tilde{M}$ and $\tilde{\Lambda} \neq \tilde{M}$, we have

$$\begin{array}{ccc} \tilde{\Lambda} \geq \tilde{M} & \Longrightarrow & d_{\bar{0}}(\tilde{\Lambda}) \geq d_{\bar{0}}(\tilde{M}) \ , \\ \tilde{\Lambda} > \tilde{M} & \Longrightarrow & d_{\bar{0}}(\tilde{\Lambda}) > d_{\bar{0}}(\tilde{M}) \ . \end{array} \tag{21}$$

Another important observation is that when the inequality (19) does not hold, then the Dynkin labels $l_1,\ldots,l_r$ of Λ must satisfy certain supplementary conditions which can be shown to imply that the representation of $\mathfrak{g}$ characterized by the highest weight Λ is atypical; see below. As we are only interested in typical representations, we may therefore impose the inequality (19) and make use of the monotonicity property (21) to provide lower bounds for the expression in equation (18). There is also an abstract argument to show that the function $d_{\bar{0}}$ continues to take integer values as long as $2(\rho_1,\alpha_s^0)/(\alpha_s^0,\alpha_s^0)$ is an integer, due to the following

Proposition: Let P be a polynomial of degree r in one real variable which takes integer values on all integers greater than some fixed integer. Then P takes integer values on all integers.

> *Proof:* The basic trick for the proof is to expand the polynomial P not in the standard basis of polynomials x^l $(l = 0, 1, \ldots, r)$ but in a different basis of polynomials defined by the binomial coefficients, that is, to write
>
> $$P(x) \;=\; \sum_{l=0}^{r} a_l \binom{x}{l} \;=\; \sum_{l=0}^{r} \frac{a_l}{l!}\, x\,(x-1)\ldots(x-l+1)~. \tag{22}$$
>
> Observing that
>
> $$\binom{x+1}{l} - \binom{x}{l} \;=\; \binom{x}{l-1}$$
>
> and therefore
>
> $$P(x+1) \,-\, P(x) \;=\; \sum_{k=0}^{r-1} a_{k+1} \binom{x}{k}~,$$
>
> we may conclude by induction on r that the property of $P(n)$ being an integer for all $n \in \mathbb{Z}$ and the – apparently weaker – property of $P(n)$ being an integer for all $n \in \mathbb{Z}$ satisfying $n \geq n_0$ for some $n_0 \in \mathbb{Z}$ are both equivalent to the fact that the coefficients a_l of P in the expansion (22) are all integers; in fact, they can be computed recursively from the formula
>
> $$\sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} P(x+i) \;=\; \sum_{k=0}^{r-p} a_{k+p} \binom{x}{k}~, \tag{23}$$
>
> which in turn can be inferred from the previous one by induction on p.

According to Table 2, this implies that the only type II Lie superalgebras for which $d_{\bar{0}}$ may take non-integer values and hence $\dim V_\Lambda$ need no longer be a multiple of 2^{N_1} are those belonging to the series $B(m\,|\,n) = \mathfrak{osp}(2m+1\,|\,2n)$ $(m, n \geq 1)$, those belonging to the series $B(0\,|\,n) = \mathfrak{osp}(1\,|\,2n)$ $(n \geq 1)$ and, finally, the exceptional Lie superalgebra $G(3)$.

With these generalities out of the way, we can proceed to determine the typical codon representations, that is, the 64-dimensional irreducible representations, of basic classical Lie superalgebras. For type I Lie superalgebras, this is easily done by exploiting the dimension formula (14), which implies that the number N_1 of positive odd roots must not exceed 6 and that Λ must be the highest weight of an irreducible representation of $\mathfrak{g}_{\bar{0}}$ of dimension $2^{\,6-N_1}$:

- The series $\mathfrak{sl}(m+1\,|\,n+1)$ with $m > n \geq 0$:
  Here, N_1 equals $(m+1)(n+1)$, so we must have $m \leq 2$, $n \leq 1$, which leaves the following possibilities:
  either $n = 0$ and $m = 0, 1, 2, 3, 4, 5$,
  or $n = 1$ and $m = 2$.

- The series $\mathfrak{sl}(n+1\,|\,n+1)$ with $n \geq 1$:
  Here, N_1 equals $(n+1)^2$, so we must have $n = 1$.

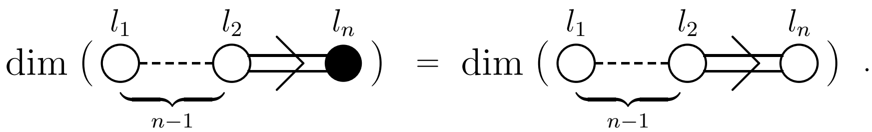
- The series $\mathfrak{osp}(2\,|\,2n)$ with $n \geq 2$:
  Here, N_1 equals $2n$, so we must have $n \leq 3$.

This leads to the list of typical codon representations of type I Lie superalgebras presented in Table 3. Note that the coefficient l_s of Λ along the simple odd root α_s remains unspecified: it can take any complex value except 0 and a few other integers that must be excluded in order to guarantee that the representation is indeed typical, and its choice has no influence on the dimension of the representation.

Table 3: Typical codon representations of type I Lie superalgebras

Lie Superalgebra	N_1	Highest Weight of $\mathfrak{g}$	Highest Weight of $\mathfrak{g}_{\bar{0}}$	Typicality Condition
$\mathfrak{sl}(2\,\|\,1)$	2	$(15, l_2)$	15	$-l_2 \neq 0, 16$
$\mathfrak{sl}(3\,\|\,1)$	3	$(1, 1, l_3)$	$(1, 1)$	$-l_3 \neq 0, 2, 4$
$\mathfrak{sl}(4\,\|\,1)$	4	$(1, 0, 0, l_4)$ $(0, 0, 1, l_4)$	$(1, 0, 0)$ $(0, 0, 1)$	$-l_4 \neq 0, 1, 2, 4$ $-l_4 \neq 0, 2, 3, 4$
$\mathfrak{sl}(6\,\|\,1)$	6	$(0, 0, 0, 0, 0, l_6)$	$(0, 0, 0, 0, 0)$	$-l_6 \neq 0, 1, 2, 3, 4, 5$
$\mathfrak{sl}(2\,\|\,2)$	4	$(3, l_2, 0)$ $(1, l_2, 1)$ $(0, l_2, 3)$	$(3) - (0)$ $(1) - (1)$ $(0) - (3)$	$l_2 \neq -4, -3, 0, 1$ $l_2 \neq -2, 0, 2$ $l_2 \neq -1, 0, 3, 4$
$\mathfrak{sl}(3\,\|\,2)$	6	$(0, 0, l_3, 0)$	$(0, 0) - (0)$	$l_3 \neq -2, -1, 0, 1$
$\mathfrak{osp}(2\,\|\,4)$	4	$(l_1, 1, 0)$	$(1, 0)$	$l_1 \neq 0, 2, 4, 6$
$\mathfrak{osp}(2\,\|\,6)$	6	$(l_1, 0, 0, 0)$	$(0, 0, 0)$	$l_1 \neq 0, 1, 2, 4, 5, 6$

For type II Lie superalgebras, the analysis can be carried out along similar lines. To begin with, we exclude the series $B(0\,|\,n) = \mathfrak{osp}(1\,|\,2n)$ $(n \geq 1)$, since it does not provide any 64-dimensional irreducible representations. This can be derived from the remarkable fact [17] that the irreducible representations of the type II Lie superalgebra $B(0\,|\,n) = \mathfrak{osp}(1\,|\,2n)$ (which by the way is the only one for which all irreducible representations are typical) are in one-to-one correspondence with those irreducible representations of the ordinary simple Lie algebra $B_n = \mathfrak{so}(2n+1)$ for which the last Dynkin label, i.e., the coefficient l_n associated with the short simple root, is even – a correspondence that can be represented graphically in the form



(24)

Note that there is no change in the Dynkin labels, so that according to the integrality condition on the Dynkin label (9), l_n must be even, since for the $B(0,n)$ series, $s = n$, $\alpha_n^0 = 2\alpha_n$ and $l_n^0 = \frac{1}{2}\,l_n$. But it is known from evaluation of the standard Weyl dimension formula that the only 64-dimensional irreducible representations of the B_n-series occur for $B_1 = \mathfrak{so}(3)$, with highest weight 63, for $B_2 = \mathfrak{so}(5)$, with highest weight $(1,3)$, and for $B_6 = \mathfrak{so}(13)$, with highest weight $(0,0,0,0,0,1)$. For the remaining type II Lie superalgebras, we argue case by case, as follows.

- The series $B(m\,|\,n) = \mathfrak{osp}(2m+1\,|\,2n)$ with $m,n \geq 1$:
  For $\mathfrak{g} = \mathfrak{osp}(2m+1\,|\,2n)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n)$, $r = m+n$, $s = n$ and $N_1 = (2m+1)n$, so equation (18) takes the form

$$\begin{aligned} \dim V_\Lambda \;=\; 2^{(2m+1)n} \,&\times\, d_{\mathfrak{sp}(2n)}(l_1, \ldots, l_{n-1}, \tilde{l}_n^0) \\ &\times\, d_{\mathfrak{so}(2m+1)}(l_{n+1}, \ldots, l_{n+m-1}, l_{n+m})~, \end{aligned} \tag{25}$$

  where

$$l_n^0 \;=\; l_n \,-\, \Big(l_{n+1} \,+ \ldots +\, l_{n+m-1} \,+\, \tfrac{1}{2}\,l_{n+m}\Big)~, \tag{26}$$

  and

$$\tilde{l}_n^0 \;=\; l_n^0 \,-\, m \,-\, \tfrac{1}{2}~. \tag{27}$$

  If $l_n^0 < m$, write $l_n^0 = k-1$ where $1 \leq k \leq m$; then the supplementary conditions [15, pp. 251/252] require that

$$l_{n+k} \;=\; \ldots \;=\; l_{n+m} \;=\; 0~,$$

  and this forces $\Lambda + \rho$ to be orthogonal to the odd root $\epsilon_n^1 - \epsilon_k^2$ [15, pp. 513-521]. Similarly, if $l_n^0 = m$ and we require in addition that $l_{n+m} = 0$, then $\Lambda + \rho$ will be orthogonal to the odd root $\epsilon_n^1 + \epsilon_m^2$ [15, pp. 513-521].

In both cases, this implies that the representation of $\mathfrak{g}$ characterized by the highest weight Λ is atypical. Thus we may assume that $l_n^0 \geq m$ and use the monotonicity property (21), distinguishing two cases:

$l_n^0 > m$**:** In this case,

$$\begin{aligned} \dim V_\Lambda \;&\geq\; 2^{(2m+1)n}\, d_{\mathfrak{sp}(2n)}(0,\ldots,0,\tfrac{1}{2})\, d_{\mathfrak{so}(2m+1)}(0,\ldots,0,0) \\ &=\; 2^{2mn} \binom{2n+1}{n}\,. \end{aligned}$$

$l_n^0 = m$**:** In this case,

$$\begin{aligned} \dim V_\Lambda \;&\geq\; 2^{(2m+1)n}\, d_{\mathfrak{sp}(2n)}(0,\ldots,0,-\tfrac{1}{2})\, d_{\mathfrak{so}(2m+1)}(0,\ldots,0,1) \\ &=\; 2^{m(2n+1)}\,. \end{aligned}$$

In both cases, we conclude that $\dim V_\Lambda$ will exceed 64 except when $m=1$ and $n \leq 2$ or when $m \leq 2$ and $n = 1$.

- The series $D(m\,|\,n) = \mathfrak{osp}(2m\,|\,2n)$ with $m \geq 2$ and $n \geq 1$:
  For $\mathfrak{g} = \mathfrak{osp}(2m\,|\,2n)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{so}(2m) \oplus \mathfrak{sp}(2n)$, $r = m+n$, $s = n$ and $N_1 = 2mn$, so equation (18) takes the form

$$\begin{aligned} \dim V_\Lambda \;=\; 2^{2mn} \;&\times\; d_{\mathfrak{sp}(2n)}(l_1,\ldots,l_{n-1},\tilde{l}_n^0) \\ &\times\; d_{\mathfrak{so}(2m)}(l_{n+1},\ldots,l_{n+m-2},l_{n+m-1},l_{n+m})\;, \end{aligned} \tag{28}$$

where

$$l_n^0 \;=\; l_n - \Big(l_{n+1} + \ldots + l_{n+m-2} + \tfrac{1}{2}\,(l_{n+m-1} + l_{n+m})\Big)\;, \tag{29}$$

and

$$\tilde{l}_n^0 \;=\; l_n^0 - m\;. \tag{30}$$

If $l_n^0 < m$, write $l_n^0 = k-1$ where $1 \leq k \leq m$; then the supplementary conditions [15, pp. 251/252] require that

$$\begin{aligned} l_{n+k} \;=\; \ldots \;=\; l_{n+m} \;=\; 0 \qquad &\text{if } l_n^0 < m-1\;, \\ l_{n+m-1} \;=\; l_{n+m} \qquad &\text{if } l_n^0 = m-1\;, \end{aligned}$$

and this forces $\Lambda + \rho$ to be orthogonal to the odd root $\epsilon_n^1 - \epsilon_k^2$ [15, pp. 525-532]. Similarly, if $l_n^0 = m$ and we require in addition that $l_{n+m-1} = 0$ and $l_{n+m} = 0$, then $\Lambda + \rho$ will be orthogonal to the odd root $\epsilon_n^1 + \epsilon_{m-1}^2$ [15, pp. 525-532]. In both cases, this implies that the representation of $\mathfrak{g}$ characterized by the highest weight Λ is atypical. Thus we may assume that $l_n^0 \geq m$ and use the monotonicity property (21), distinguishing two cases:

$l_n^0 > m$**:** In this case,

$$\begin{aligned}\dim V_\Lambda \ &\geq\ 2^{2mn}\ d_{\mathfrak{sp}(2n)}(0,\ldots,0,1)\ d_{\mathfrak{so}(2m)}(0,\ldots,0,0,0)\\ &=\ 2^{2mn+1}\,\frac{1}{n}\binom{2n+1}{n-1}\ .\end{aligned}$$

$l_n^0 = m$**:** In this case, $l_{n+m-1} > 0$ and

$$\begin{aligned}\dim V_\Lambda \ &\geq\ 2^{2mn}\ d_{\mathfrak{sp}(2n)}(0,\ldots,0,0)\ d_{\mathfrak{so}(2m)}(0,\ldots,0,1,0)\\ &=\ 2^{m(2n+1)-1}\ ,\end{aligned}$$

or $l_{n+m} > 0$ and

$$\begin{aligned}\dim V_\Lambda \ &\geq\ 2^{2mn}\ d_{\mathfrak{sp}(2n)}(0,\ldots,0,0)\ d_{\mathfrak{so}(2m)}(0,\ldots,0,0,1)\\ &=\ 2^{m(2n+1)-1}\ .\end{aligned}$$

In both cases, we conclude that $\dim V_\Lambda$ will exceed 64 except when $m = 2$ and $n = 1$.

- The family $D(2\,|\,1\,;\alpha) = \mathfrak{osp}(4\,|\,2\,;\alpha)$ with $\alpha \neq 0,-1,\infty$:
  For $\mathfrak{g} = \mathfrak{osp}(4\,|\,2\,;\alpha)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)$, $r = 3$, $s = 1$ and $N_1 = 4$, so equation (18) takes the form

$$\begin{aligned}\dim V_\Lambda \ &=\ 16\ d_{\mathfrak{su}(2)}(\tilde{l}_1^0)\ d_{\mathfrak{su}(2)}(l_2)\ d_{\mathfrak{su}(2)}(l_3) &(31)\\ &=\ 16\ (1+\tilde{l}_1^0)\,(1+l_2)\,(1+l_3)\ , &(32)\end{aligned}$$

where

$$l_1^0 \ =\ l_1 \ -\ \tfrac{1}{2}\,(l_2+l_3)\ , \tag{33}$$

and

$$\tilde{l}_1^0 \ =\ l_1^0 \ -\ 2\ . \tag{34}$$

If $l_1^0 < 2$, the supplementary conditions [15, pp. 251/252] require that

$$\begin{aligned}l_2 \ =\ l_3 \ &=\ 0 &&\text{if } l_1^0 = 0\ ,\\ \alpha\,(l_3+1)\ &=\ l_2+1 &&\text{if } l_1^0 = 1\ ,\end{aligned}$$

and this forces $\Lambda+\rho$ to be orthogonal to the simple odd root α_1 in the first case and to the odd root $\alpha_1+\alpha_2$ in the second case [15, pp. 532-537], which implies that the representation of $\mathfrak{g}$ characterized by the highest weight Λ is atypical. Thus we may assume that $l_1^0 \geq 2$.

- The algebra $F(4)$:

  For $\mathfrak{g} = F(4)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{su}(2) \oplus \mathfrak{so}(7)$, $r = 4$, $s = 1$ and $N_1 = 8$, so equation (18) takes the form

$$\begin{aligned} \dim V_\Lambda &= 256\, d_{\mathfrak{su}(2)}(\tilde{l}_1^0)\, d_{\mathfrak{so}(7)}(l_4, l_3, l_2) && (35) \\ &= 256\, (1 + \tilde{l}_1^0)\, d_{\mathfrak{so}(7)}(l_4, l_3, l_2) \ , && (36) \end{aligned}$$

  where

$$l_1^0 = \tfrac{1}{3}\,(2l_1 - 3l_2 - 4l_3 - 2l_4) \ , \qquad (37)$$

  and

$$\tilde{l}_1^0 = l_1^0 - 4 \ . \qquad (38)$$

  If $l_1^0 < 4$, the supplementary conditions [15, pp. 251/252] require that $l_1^0 \neq 1$ and

$$\begin{aligned} l_2 = l_3 = l_4 &= 0 \qquad \text{if } l_1^0 = 0 \ , \\ l_2 = l_4 &= 0 \qquad \text{if } l_1^0 = 2 \ , \\ l_2 &= 2l_4 + 1 \qquad \text{if } l_1^0 = 3 \ , \end{aligned}$$

  and this forces $\Lambda + \rho$ to be orthogonal to the simple odd root α_1 in the first case, to the odd root $\alpha_1 + \alpha_2 + \alpha_3$ in the second case and to the odd root $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ in the third case [15, pp. 537-541], which implies that the representation of $\mathfrak{g}$ characterized by the highest weight Λ is atypical. Thus we may assume that $l_1^0 \geq 4$ and deduce that the the dimension of any typical representation of $F(4)$ is a multiple of 256.

- The algebra $G(3)$:

  For $\mathfrak{g} = G(3)$, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{su}(2) \oplus G_2$, $r = 3$, $s = 1$ and $N_1 = 7$, so equation (18) takes the form

$$\begin{aligned} \dim V_\Lambda &= 128\, d_{\mathfrak{su}(2)}(\tilde{l}_1^0)\, d_{G_2}(l_3, l_2) && (39) \\ &= 128\, (1 + \tilde{l}_1^0)\, d_{G_2}(l_3, l_2) \ , && (40) \end{aligned}$$

  where

$$l_1^0 = \tfrac{1}{2}\,(l_1 - 2l_2 - 3l_3) \ , \qquad (41)$$

  and

$$\tilde{l}_1^0 = l_1^0 - \tfrac{7}{2} \ . \qquad (42)$$

  If $l_1^0 < 3$, the supplementary conditions [15, pp. 251/252] require that $l_1^0 \neq 1$ and

$$\begin{aligned} l_2 = l_3 &= 0 \qquad \text{if } l_1^0 = 0 \ , \\ l_2 &= 0 \qquad \text{if } l_1^0 = 3 \ , \end{aligned}$$

and this forces $\Lambda+\rho$ to be orthogonal to the simple odd root α_1 in the first case and to the odd root $\alpha_1+\alpha_2+\alpha_3$ in the second case [15, pp. 542-545]. Similarly, if $l_1^0 = 3$ and we require in addition that $l_2 = 0$ and $l_3 = 0$, then $\Lambda + \rho$ will be orthogonal to the odd root $\alpha_1 + 3\alpha_2 + \alpha_3$ [15, pp. 542-545]. In both cases, this implies that the representation of $\mathfrak{g}$ characterized by the highest weight Λ is atypical. Thus we may assume that $l_1^0 \geq 3$ and deduce that the dimension of any typical representation of $G(3)$ is a multiple of 64; moreover, the only candidate of dimension equal to 64 ($l_1 = 6$, $l_2 = 0$, $l_3 = 0$) is excluded, because it is atypical.

With these restrictions, it is now an easy exercise to write down the highest weights of all irreducible representations of $\mathfrak{g}_{\bar{0}}$ of the correct dimension and to eliminate all candidates that fail to satisfy the typicality conditions; the result is presented in Table 4. Note that in the family $D(2\,|\,1;\alpha) = \mathfrak{osp}(4\,|\,2\,;\alpha)$, the parameter α remains unspecified: it can take any complex value except 0, -1, ∞ and a few other rational numbers that must be excluded in order to guarantee that the representation is indeed typical, and its choice has no influence on the dimension of the representation.

Table 4: Typical codon representations of type II Lie superalgebras

Lie Superalgebra	Highest Weight of $\mathfrak{g}$	Highest Weight of $\mathfrak{g}_{\bar{0}}$	Typicality Condition
$\mathfrak{osp}(3\,|\,2)$	$(\frac{17}{2}, 15)$	$(1) - (15)$	
$\mathfrak{osp}(5\,|\,2)$	$(\frac{5}{2}, 0, 1)$	$(2) - (0,1)$	
$\mathfrak{osp}(3\,|\,4)$	$(0, \frac{5}{2}, 3)$	$(0,1) - (3)$	
$\mathfrak{osp}(4\,|\,2\,;\alpha)$	$(\frac{1}{2}\,(5\alpha+5), 0, 0)$	$(5) - (0) - (0)$	$\alpha \neq -\frac{5}{3}, -\frac{3}{5}$
	$(\frac{1}{2}\,(3\alpha+4), 1, 0)$	$(3) - (1) - (0)$	$\alpha \neq -4, -\frac{4}{3}$
	$(\frac{1}{2}\,(4\alpha+3), 0, 1)$	$(3) - (0) - (1)$	$\alpha \neq -\frac{1}{4}, -\frac{3}{4}$
	$(\frac{1}{2}\,(3\alpha+3), 1, 1)$	$(2) - (1) - (1)$	$\alpha \neq 3, -\frac{1}{3}$
	$(\frac{1}{2}\,(2\alpha+5), 3, 0)$	$(2) - (3) - (0)$	$\alpha \neq -\frac{5}{2}, \frac{3}{2}$
	$(\frac{1}{2}\,(5\alpha+2), 0, 3)$	$(2) - (0) - (3)$	$\alpha \neq -\frac{2}{5}, \frac{2}{3}$

4 Conclusions and Outlook

The main result of the present paper, the first in a sequence of two, is the complete list of all typical codon representations (typical 64-dimensional irreducible representations) of basic classical Lie superalgebras, presented in Table 3 and Table 4: we find 12 basic classical Lie superalgebras with a total of 18 codon representations that are essentially different (conjugate representations are not regarded as essentially different). The analysis is based on the classification of basic classical Lie superalgebras and on their representation theory, which are briefly reviewed in Sect. 2 and Sect. 3, respectively, in particular on the Weyl-Kac dimension formula for typical representations. As in the case of the ordinary Weyl dimension formula for irreducible representations of ordinary simple Lie algebras, the dimension of the representation grows with its highest weight, so that no algebra belonging to any of the classical series will, from a certain rank upwards, admit codon representations (or, more generally, non-trivial representations of dimension ≤ 64) at all. The main difficulty to be overcome was to extend this monotonicity argument from ordinary simple Lie algebras to basic classical Lie superalgebras and to derive sufficiently sharp lower bounds on dimensions of typical representations, in order to exclude the appearance of algebras of higher rank. In this respect, the final results are more stringent for basic classical Lie superalgebras than they are for ordinary simple Lie algebras.

On the other hand, it must be stressed that for atypical representations, a general dimension formula is still not known, and this is a major obstacle to performing a similar analysis for this kind of representations – despite the fact that there is no reason to regard typical representations as being more important than atypical ones; see [15, p. 258/259] for comments on this matter. Similarly, the existence and classification of codon representations of the strange classical Lie superalgebras is an open problem. In this sense, the analysis presented in the present paper is not complete.

Despite these limitations, our investigation does provide a framework for the subsequent investigation of branching schemes, the main goal being to identify the ones that reproduce the standard genetic code. This analysis will be performed in the forthcoming second paper of this series.

Acknowledgments

The authors would like to thank Prof. J.E.M. Hornos for his incentive and support of the present project, Prof. A. Sciarrino and Prof. P. Jarvis for clarifying correspondence on the representation theory of Lie superalgebras and Prof. A. Grishkov for fruitful discussions.

References

[1] J.E.M. Hornos and Y.M.M. Hornos: *Algebraic Model for the Evolution of the Genetic Code*, Phys. Rev. Lett. **71** (1993) 4401-4404.

[2] W.G. McKay and J. Patera: *Tables of Dimensions, Indices and Branching Rules for Representations of Simple Lie Algebras*, Lecture Notes in Pure and Applied Mathematics, Vol. 69, Marcel Dekker, New York (1981).

[3] J. Maddox: *The Genetic Code by Numbers*, Nature **367** (1994) 111.

[4] I. Stewart: *Broken Symmetry in the Genetic Code?*, New Scientist **141**, No. 1915 (1994) 16.

[5] M. Forger: *Symmetry Breaking in the Genetic Code*, Minicourse given at the 41^o Seminário Brasileiro de Análise, Campinas, May 1995.

[6] F.H.C. Crick: *The Origin of the Genetic Code*, J. Mol. Biol. **38** (1968) 367-379.

[7] T.H. Jukes: *Evolution of the Amino Acid Code: Inferences from Mitochondrial Codes*, J. Mol. Evol. **19** (1983) 219-225.

[8] S. Osawa, T.H. Jukes, K. Watanabe and A. Muto: *Recent Evidence for Evolution of the Genetic Code*, Microbiol. Rev. **56** (1992) 229-264.

[9] F.F. Ferreira: *Construção das Representações Irredutíveis das Álgebras q-Deformadas $U_q(sl(2))$ e $U_q(sl(3))$ em Raíses da Unidade*, Master Thesis, Institute of Physics, University of São Paulo, São Carlos, 1997.

[10] J.D. Bashford, I. Tsohantjis and P.D. Jarvis: *Supersymmetry and the Evolution of the Genetic Code*, Preprint UTAS-PHYS-96-02, *Codon and Nucleotide Assignments in a Supersymmetric model of the Genetic Code*, Phys. Lett. **A 233** (1997) 481-488.

[11] L. Frappat, P. Sorba and A. Sciarrino: *A Crystal Base for the Genetic Code*, Preprint ENSLAPP-AL-671/97, DSF-97/37, physics/9801027, December 1997.

[12] V.G. Kac: *Lie Superalgebras*, Adv. Math. **26** (1977) 8-96.

[13] V.G. Kac: *Representations of Classical Lie Superalgebras*, in: Proceedings of the VIth International Conference on Differential Geometric Methods in Theoretical Physics, Bonn, Germany (1977), Lecture Notes in Mathematics, Vol. 676, pp. 597-626, Springer-Verlag, Berlin (1978).

[14] M. Scheunert: *The Theory of Lie Superalgebras*, Lecture Notes in Mathematics, Vol. 716, Springer-Verlag, Berlin (1979).

[15] J.F. Cornwell: *Group Theory in Physics*, Vol. III, Academic Press, New York 1989.

[16] L. Frappat, P. Sorba and A. Sciarrino: *Dictionary on Lie Superalgebras*, Preprint ENSLAPP-AL-600/96, DSF-T-30/96, hep-th/9607161, July 1996.

[17] V. Rittenberg and M. Scheunert: *A Remarkable Connection Between the Representations of the Lie Superalgebras* $\mathfrak{osp}(1\,|\,2N)$ *and the Lie Algebras* $\mathfrak{o}(2N+1)$, Commun. Math. Phys. **83** (1982) 1-9.